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On the Phase Transition to Sheet Percolation in Random Cantor Sets

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The d-dimensional random Cantor set is a generalization of the classical "middle-thirds" Cantor set. Starting with the unit cube $[0, 1]^d$, at every stage of the construction we divide each cube remaining into M^d equal subcubes, and select each of these at random with probability p. The resulting limit set is a random fractal, which may be crossed by paths or $(d-1)$-dimensional "sheets". We examine the critical probability $p_s(M, d)$ marking the existence of these sheet crossings, and show that $p_s(M, d) \to 1 - p_c(\mathbb{M}^d)$ as $M \to \infty$, where $p_c(\mathbb{M}^d)$ is the critical probability of site percolation on the lattice $\mathbb{M}^d$ obtained by adding the diagonal edges to the hypercubic lattice $\mathbb{Z}^d$. This result is then used to show that, at least for sufficiently large values of M, the phases corresponding to the existence of path and sheet crossings are distinct.

KEY WORDS: Random Cantor sets; fractal percolation; critical probability.

1. INTRODUCTION

We consider the fractal percolation process first proposed by Mandelbrot[12] and subsequently studied by several authors. In this section we briefly review their work and present our main results. Let $d \geqslant 2$, $M \geqslant 2$, and $0 < p < 1$. We construct the "d-dimensional random Cantor set" $C^{[M]}$ as follows. Write C_0 for the unit cube $[0, 1]^d$ of $\mathbb{R}^d$. Divide C_0 into M^d equal closed subcubes, each of side length M^{-1}, in the natural way. Select each of these subcubes independently with probability p, and write C_1 for the union of these *level-1* cubes thus selected. Similarly, divide each cube of C_1 into M^d subcubes each of side length M^{-2} and select each of these independently with probability p, writing C_2 for the union of these *level-2* cubes. Continuing this process, we obtain a decreasing sequence of closed

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sets $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, with limit $C^{[M]} = \bigcap_{n=0}^{\infty} C_n$. We shall normally drop the superscript when M is fixed, and write the limit set as just C.

Provided that the process does not become extinct (that is, provided that $C_n \neq \emptyset$ for all n), the set C is a random fractal which may be in one of several "phases" as characterized by Dekking and Meester.[5] In particular, we define *path-percolation* to occur in a set S if S contains a connected component intersecting both the "left-hand face" $L = \{0\} \times [0, 1]^{d-1}$ and the "right-hand face" $R = \{1\} \times [0, 1]^{d-1}$ of C_0. Chayes *et al.*[3] demonstrate the existence of a nontrivial phase transition to path-percolation in the set C as we vary the value of p, that is, there exists a critical probability $p_c(M, d)$ with $0 < p_c(M, d) < 1$ such that if $p > p_c(M, d)$, then path-percolation occurs with positive probability in C, while if $p < p_c(M, d)$, then path-percolation almost surely does not occur in C. [In fact, at least if $d = 2$, path-percolation occurs with positive probability in C whenever $p \geqslant p_c(M, d)$.] Meester[13] gave an alternative definition of percolation in terms of arcwise-connected components, and showed this to be probabilistically equivalent to the notion of path-percolation above.

Chayes and Chayes[2] considered the behavior of the critical probability $p_c(M, 2)$ for large values of M. They proved that

$$p_c(M, 2) \to p_c(\mathbb{Z}^2) \qquad \text{as} \quad M \to \infty \tag{1.1}$$

where $p_c(\mathbb{Z}^2)$ denotes the critical probability for site percolation on the ordinary square lattice with vertex set $\mathbb{Z}^2$. See Grimmett[9] for a general account of percolation theory on this and other lattices.

Falconer and Grimmett[6,7] generalized this result to $d \geqslant 2$ in the following way. Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and edge set given by the adjacency relation: $\mathbf{x} \sim \mathbf{y}$ if and only if $|x_i - y_i| \leqslant 1$ for all i, and $x_i = y_i$ for at least one value of i, where $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$. When $d = 2$, $\mathbb{L}^2$ is the usual square lattice $\mathbb{Z}^2$ as above. If $d \geqslant 3$, then $\mathbb{L}^d$ contains the d-dimensional hypercubic lattice $\mathbb{Z}^d$ as a strict sublattice. They concluded that

$$p_c(M, d) \to p_c(\mathbb{L}^d) \qquad \text{as} \quad M \to \infty \tag{1.2}$$

where $p_c(\mathbb{L}^d)$ denotes the critical probability for site percolation on the lattice $\mathbb{L}^d$.

When $d \geqslant 3$, we may also consider the existence of $(d-1)$-dimensional "sheets" crossing C. We define *sheet-percolation* to occur in a set S if S contains a surface separating the left-hand face L and the right-hand face R of C_0. It will be easier to work with the complementary set $S^c = [0, 1]^d \backslash S$ and observe that sheet-percolation occurs in S if and only if S^c does not contain a continuous path γ: $[0, 1] \to S^c$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$.

We define $p_s(M, d) = \sup\{p: P(\text{sheet-percolation occurs in } C) = 0\}$. Certainly we have $p_s(M, d) \geqslant p_c(M, d) > 0$, since any surface crossing $[0, 1]^d$ contains a path crossing $[0, 1]^d$ (subject to reordering the axes). As observed by Chayes *et al.*[4] in the case $d = 3$, it is easy to show that $p_s(M, d) < 1$, by a method analogous to the case of path-percolation in two dimensions.

We now define a further d-dimensional lattice. Let $\mathbb{M}^d$ be the lattice with vertex set $\mathbb{Z}^d$ and edge set given by the adjacency relation: $\mathbf{x} \sim \mathbf{y}$ if and only if $|x_i - y_i| \leqslant 1$ for all i. Thus, for $d \geqslant 2$, $\mathbb{M}^d$ contains both the lattices $\mathbb{Z}^d$ and $\mathbb{L}^d$ as strict sublattices, and is obtained from $\mathbb{Z}^d$ by an enhancement permitting connections between "diagonally adjacent" pairs of vertices. In addition, we define the sublattice $B_N(\mathbb{M}^d)$ of $\mathbb{M}^d$ of size $N \times \cdots \times N$ to be the lattice with vertex set $\{0, 1,..., N-1\}^d$ and edges inherited from $\mathbb{M}^d$.

We study the problem of site percolation on the lattice $\mathbb{M}^d$, and let $p_c(\mathbb{M}^d)$ denote the critical probability for this process.

Theorem 1. $p_s(M, d) \geqslant 1 - p_c(\mathbb{M}^d)$ for all values of M and d.

Theorem 2. Let $p > 1 - p_c(\mathbb{M}^d)$. Then for all values of d

$$P(\text{sheet-percolation in } C^{[M]}) \to 1 \qquad \text{as} \quad M \to \infty$$

Theorem 3. For all values of d

$$p_s(M, d) \to 1 - p_c(\mathbb{M}^d) \qquad \text{as} \quad M \to \infty$$

The proof of Theorem 3 is immediate from Theorems 1 and 2.

The reader should contrast this result with (1.2). The lattice $\mathbb{M}^d$, rather than $\mathbb{L}^d$, appears because it is the existence of paths in the complement which determines whether or not sheet-percolation occurs; for this, it is sufficient for vacant cubes to meet only at a corner.

Corollary. For all $d \geqslant 3$, we have $p_c(M, d) < p_s(M, d)$ for all sufficiently large values of M.

Proof of Corollary. Combining (1.2) and Theorem 3, it is sufficient to show that

$$p_c(\mathbb{L}^d) < 1 - p_c(\mathbb{M}^d) \tag{1.3}$$

We note that $\mathbb{M}^d$ is obtained from $\mathbb{L}^d$ by an enhancement permitting extra connections between vertices, so certainly we have $p_c(\mathbb{M}^d) \leqslant p_c(\mathbb{L}^d)$. Similarly we have $p_c(\mathbb{L}^d) \leqslant p_c(\mathbb{Z}^d) < 1/2$, where the last inequality is from Campanino and Russo,[1] which is sufficient for (1.3). ∎

This corollary extends a conclusion of Chayes *et al.*,[4] proved in the special case of $d=3$ and the box $[0,2]\times[0,2]\times[0,1]$, to the unit cube $[0,1]^d$, showing that, at least for sufficiently large M, the path-percolation and sheet-percolation phases are indeed distinct phases.

Note also that when we apply Theorem 3 in the case $d=2$, the concepts of path- and sheet-percolation are identical (subject to interchanging the axes), and hence we deduce that $p_c(M, 2) \to 1 - p_c(\mathbb{M}^2)$. In conjunction with (1.1), this shows that $p_c(\mathbb{M}^2) + p_c(\mathbb{Z}^2) = 1$, an equality observed by Sykes and Essam[16] and subsequently rigorously proved by Russo[15] and Kesten.[11]

Exact values for critical probabilities of site percolation in these lattices are not known. The best known bounds for $p_c(\mathbb{Z}^2)$ are currently $0.556 < p_c(\mathbb{Z}^2) < 0.682$, the first inequality due to van den Berg and Ermakov,[17] the second due to Zuev,[18] with the exact value likely to be around 0.593.

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

2. PROOF OF THEOREM 1

To prove that $p_s(M, d) \geqslant 1 - p_c(\mathbb{M}^d)$ for all values of M and d, we show that whenever $p < 1 - p_c(\mathbb{M}^d)$, then sheet-percolation does not occur in C, almost surely. Note that from the compactness of C it follows that

$$\{\text{sheet-percolation in } C\} = \bigcap_{n=0}^{\infty} \{\text{sheet-percolation in } C_n\} \tag{2.1}$$

which is an intersection of a decreasing sequence of events, and therefore

$$P(\text{sheet-percolation in } C) = \lim_{n\to\infty} P(\text{sheet-percolation in } C_n) \tag{2.2}$$

We define another, stronger concept of percolation as follows: We say that *full sheet-percolation* occurs in a set S if the interior of S separates the left-hand face L and the right-hand face R of C_0, that is, if and only if S^*, defined by $S^* = \overline{[0,1]^d \backslash S}$, does not contain a continuous path $\gamma\colon [0,1] \to S^*$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$. Thus, we may think of a family S of level-n cubes that forms a surface separating the left- and right-hand faces of C_0 as being full if all the pairs of adjacent cubes $\{C', C''\}$ which are necessary to block paths γ in S^c have $\dim(C' \cap C'') = d-1$, that is, C' and C'' intersect in a $(d-1)$-dimensional "face" rather than an "edge" of dimension less than $(d-1)$.

Lemma 1. We have

$$P(\text{sheet-percolation in } C) = \lim_{n\to\infty} P(\text{full sheet-percolation in } C_n)$$

The proof of this lemma, which is omitted, is based upon that of Lemma 5 of Falconer and Grimmett.[7] The idea is to show that if a sheet crossing the set C_n passes through an edge of dimension less than $(d-1)$, then for some $m>n$, enough of the level-m subcubes touching that edge will be removed so as to prevent that particular crossing, almost surely.

Continuing with the proof of Theorem 1, let $p<1-p_c(\mathbb{M}^d)$ and write $q=1-p$. We consider site percolation on the lattice $\mathbb{M}^d$, with sites being declared open independently at random with probability q. Since q is greater than the critical probability for this process, we have

$$P_q(\text{the origin belongs to an infinite open cluster}) > 0 \tag{2.3}$$

where P_q is the appropriate product probability measure. Let $\theta_q(B_N(\mathbb{M}^d))$ denote the probability that there exists a path of open vertices linking the left-hand face $\{0\}\times\{0, 1,..., N-1\}^{d-1}$ and the right-hand face $\{1\}\times\{0, 1,..., N-1\}^{d-1}$ in site percolation on the lattice $B_N(\mathbb{M}^d)$. It follows from Theorem (6.125) of Grimmett[9] that there exists $\tau>0$ such that

$$\theta_q(B_N(\mathbb{M}^d)) \geqslant \tau \tag{2.4}$$

for all $N>0$.

For each $n\geqslant 1$, we define $C_n^* = \overline{[0, 1]^d \backslash C_n}$, giving an increasing sequence of closed sets $C_0^* \subseteq C_1^* \subseteq C_2^* \subseteq \cdots$, and note that full sheet-percolation occurs in C_n if and only if C_n^* does not contain a continuous path $\gamma\colon [0, 1] \to C_n^*$ such that $\gamma(0)\in L$ and $\gamma(1)\in R$.

Let $E_n = \{\text{full sheet-percolation occurs in } C_n\}$ and set $p_n = P(E_n)$. To obtain estimates on the p_n, we compare the sets C_n^* (consisting of a union of cubes of side length M^{-n}) to sublattices of $\mathbb{M}^d$ in the natural way: Open vertices of $B_{M^n}(\mathbb{M}^d)$ correspond to cubes present in C_n^*, with two vertices being considered adjacent if and only if the corresponding cubes have at least a point in common. By this comparison, conditioning on full retention at level-$(n-1)$, we find that

$$P(E_n^c \mid C_{n-1}^* = \varnothing) = \theta_q(B_{M^n}(\mathbb{M}^d)) \geqslant \tau > 0 \tag{2.5}$$

Therefore $p_0 = 1$ and

$$\begin{aligned} p_n &\leqslant \prod_{j=1}^{n} (1-\theta_q(B_{M^j}(\mathbb{M}^d)) \\ &\leqslant (1-\tau)^n \end{aligned} \tag{2.6}$$

by (2.5) for each $n \geqslant 1$, and so

$$p_n = P(\text{full sheet-percolation in } C_n) \to 0 \qquad \text{as} \quad n \to \infty \tag{2.7}$$

Applying Lemma 1, we conclude that

$$P(\text{sheet-percolation in } C) = 0 \tag{2.8}$$

as required. ∎

3. PROOF OF THEOREM 2

Let $d \geqslant 2$, $p > 1 - p_c(\mathbb{M}^d)$, and choose an $\varepsilon > 0$ such that $(1-\varepsilon)p > 1 - p_c(\mathbb{M}^d)$. We shall show that there exists $M(\varepsilon)$ such that for all $M \geqslant M(\varepsilon)$, we have

$$P(\text{sheet-percolation in } C_n) \geqslant 1 - \varepsilon \tag{3.1}$$

for all $n \geqslant 1$, and hence deduce, using (2.2) and letting $\varepsilon \to 0$, that

$$P(\text{sheet-percolation in } C^{[M]}) \to 1 \qquad \text{as} \quad M \to \infty \tag{3.2}$$

In the following proof, we shall assume that M is divisible by 5, although it will be clear that the method works for any $M \geqslant 5$, with the necessary slight modifications if M is not divisible by 5.

We adopt the following notation for labeling subcubes of $[0, 1]^d$. Let $J^d = \{0, 1,..., M-1\}^d$ and write

$$J^{d,m} = \{(\mathbf{i}_1,..., \mathbf{i}_m): \mathbf{i}_j \in J^d\}$$

setting $J^{d,0} = \{\emptyset\}$. With each *index*

$$\mathbf{I}^{(m)} = (\mathbf{i}_1,..., \mathbf{i}_m) = ((i_{1,1},..., i_{1,d}),..., (i_{m,1},..., i_{m,d})) \in J^{d,m}$$

we associate the level-m subcube $C[\mathbf{I}^{(m)}]$ of $[0, 1]^d$ given by

$$C[\mathbf{I}^{(m)}] = \mathbf{c}[\mathbf{I}^{(m)}] + [0, M^{-m}]^d$$

where

$$\mathbf{c}[\mathbf{I}^{(m)}] = \left(\sum_{j=1}^{m} M^{-j} i_{j,1},..., \sum_{j=1}^{m} M^{-j} i_{j,d}\right)$$

setting $C[\emptyset] = [0, 1]^d$.

Suppose that we are given a family $\{Z[\mathbf{I}]: \mathbf{I} \in \bigcup_{m \geq 0} J^{d,m}\}$ of independent random variables, each taking the value 1 with probability p, and 0 otherwise. For each $\mathbf{I}^{(m)} = (\mathbf{i}_1,..., \mathbf{i}_m) \in J^{d,m}$ we define the indicator function

$$1_Z[\mathbf{I}^{(m)}] = \prod_{j=1}^{m} Z[(\mathbf{i}_1,..., \mathbf{i}_j)]$$

and observe that

$$1_Z[\mathbf{I}^{(m+1)}] = 1_Z[\mathbf{I}^{(m)}]\, Z[\mathbf{I}^{(m+1)}] \tag{3.3}$$

for every $\mathbf{I}^{(m+1)} = (\mathbf{I}^{(m)}, \mathbf{i}_{m+1}) \in J^{d,m+1}$. Then by our construction, the set C_n is the union of those level-n cubes $C[\mathbf{I}^{(n)}]$ satisfying $1_Z[\mathbf{I}^{(n)}] = 1$.

Let $\mathbf{I}^{(m)} \in J^{d,m}$ and $\mathbf{k} = (k_1,..., k_d) \in \{0, 1, 2, 3, 4\}^d$. Define the level-$m$ *block* $B[\mathbf{I}^{(m)}; \mathbf{k}]$ by

$$B[\mathbf{I}^{(m)}; \mathbf{k}] = \mathbf{c}[\mathbf{I}^{(m)}] + (\tfrac{1}{5}k_1 M^{-m},..., \tfrac{1}{5}k_d M^{-m}) + [0, \tfrac{1}{5}M^{-m}]^d$$

Then each level-m cube $C[\mathbf{I}^{(m)}]$ can be written as the union of the 5^d level-m blocks contained therein, and each level-m block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is the union of $(M/5)^d$ level-$(m+1)$ subcubes of $C[\mathbf{I}^{(m)}]$.

Define the *annulus* $A[\mathbf{I}^{(m)}; \mathbf{k}]$ around a block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ by

$$A[\mathbf{I}^{(m)}; \mathbf{k}] = \{\mathbf{c}[\mathbf{I}^{(m)}] + (\tfrac{1}{5}k_1 M^{-m},..., \tfrac{1}{5}k_d M^{-m}) + [-\tfrac{1}{5}M^{-m}, \tfrac{2}{5}M^{-m}]^d\} \backslash \mathrm{int}\, B[\mathbf{I}^{(m)}; \mathbf{k}]$$

so that $A[\mathbf{I}^{(m)}; \mathbf{k}]$ is composed of the $3^d - 1$ level-m blocks touching $B[\mathbf{I}^{(m)}; \mathbf{k}]$ (or notional blocks outside $[0, 1]^d$ if $B[\mathbf{I}^{(m)}; \mathbf{k}]$ intersects the boundary of $[0, 1]^d$). Note that, with our definitions, no extra difficulties will arise with those annuli not completely contained within $[0, 1]^d$. In addition, we define $\partial^{(i)}A[\mathbf{I}^{(m)}; \mathbf{k}]$ and $\partial^{(o)}A[\mathbf{I}^{(m)}; \mathbf{k}]$ to be respectively the inner and outer components of the boundary of $A[\mathbf{I}^{(m)}; \mathbf{k}]$.

Fix $n \geq 1$. For every $m \leq n$, we now define the notions of goodness and availability for each level-m subcube $C[\mathbf{I}^{(m)}]$, $\mathbf{I}^{(m)} \in J^{d,m}$, inductively on $m = n, n-1,..., 0$ as follows:

$m = n$: We declare all level-n cubes $C[\mathbf{I}^{(n)}]$, $\mathbf{I}^{(n)} \in J^{d,n}$, to be *good*, and declare $C[\mathbf{I}^{(n)}]$ to be *available* if $Z[\mathbf{I}^{(n)}] = 1$.

$m < n$: Suppose that we have determined the availability of $C[\mathbf{I}]$ for all $\mathbf{I} \in J^{d,m+1} \cup \cdots \cup J^{d,n}$. Given subsets D, E, and S of $[0, 1]^d$, we say that S contains a *full sheet separating D and E* if there is no continuous path

$\gamma: [0, 1] \to \overline{[0, 1]^d \backslash S}$ such that $\gamma(0) \in D$ and $\gamma(1) \in E$. We say that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is *isolated* if the set

$$S = \bigcup \{C[\tilde{\mathbf{I}}^{(m+1)}]: \tilde{\mathbf{I}}^{(m+1)} \in J^{d,m+1} \text{ and } C[\tilde{\mathbf{I}}^{(m+1)}] \text{ is available}\}$$
$$\cup \{\mathbb{R}^d \backslash [0, 1]^d\}$$

contains a full sheet separating $\partial^{(i)} A[\mathbf{I}^{(m)}; \mathbf{k}]$ and $\partial^{(o)} A[\mathbf{I}^{(m)}; \mathbf{k}]$. Figure 1 illustrates an isolated block when $d = 2$.

For subsets X, Y of $\mathbb{R}^d$, we define $\text{dist}(X, Y) = \inf\{d(\mathbf{x}, \mathbf{y}): \mathbf{x} \in X, \mathbf{y} \in Y\}$, where $\mathbf{x} = (x_1,..., x_d)$, $\mathbf{y} = (y_1,..., y_d)$, and $d(\mathbf{x}, \mathbf{y}) = \max_{1 \leqslant i \leqslant d} |x_i - y_i|$, with the convention that $\inf\{\varnothing\} = \infty$.

Let $\mathbf{I}^{(m)}(1)$, $\mathbf{I}^{(m)}(2)$,..., $\mathbf{I}^{(m)}(M^{dm})$ be some fixed ordering of $J^{d,m}$. Using this ordering, we determine the goodness of each $C[\mathbf{I}^{(m)}(j)]$, $1 \leqslant j \leqslant M^{dm}$, in turn as follows: For each $1 \leqslant j \leqslant M^{dm}$, let

$$P(j) = \bigcup_{l<j} \{C[\mathbf{I}^{(m)}(l)]: C[\mathbf{I}^{(m)}(l)] \text{ is not good}\}$$

be the set of level-m cubes preceding $C[\mathbf{I}^{(m)}(j)]$ that have been examined and found to be not good. We declare the cube $C[\mathbf{I}^{(m)}(j)]$ to be *good* if $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$ is isolated for every $\mathbf{k} \in \{0,..., 4\}^d$ such that

$$\text{dist}(B[\mathbf{I}^{(m)}(j); \mathbf{k}], P(j)) \geqslant \tfrac{2}{5} M^{-m}$$

In addition, we declare the cube $C[\mathbf{I}^{(m)}(j)]$ to be *available* if it is both good and $Z[\mathbf{I}^{(m)}(j)] = 1$.

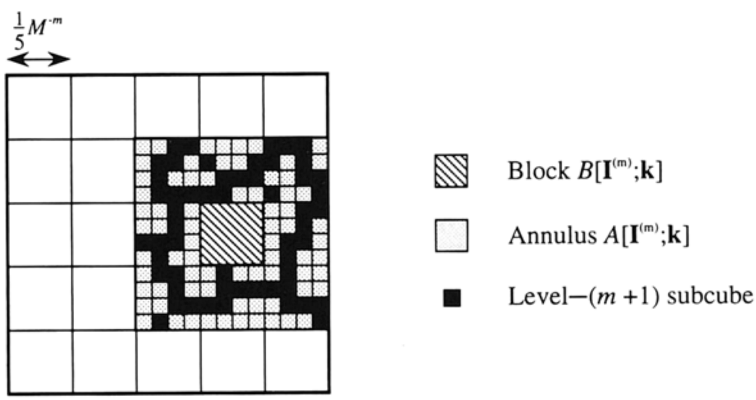

Fig. 1. A level-m cube $C[\mathbf{I}^{(m)}]$ containing an isolated block $B[\mathbf{I}^{(m)}; \mathbf{k}]$.

Informally, we have defined a level-m cube $C[\mathbf{I}^{(m)}]$ to be good if it contains a favorable arrangement of "full sheets" of smaller cubes (the exact arrangement required depending upon the status of the level-m cubes previously examined), and to be available if it is both good and retained for the next level of the inductive definition. Where convenient, we shall use the indicator function $1_A[\mathbf{I}^{(m)}]$, taking the value 1 if $C[\mathbf{I}^{(m)}]$ is available and 0 otherwise.

Using this procedure, we can determine the goodness of the level-0 cube $C[\varnothing]=[0,1]^d$.

Lemma 2. $\{C[\varnothing]$ is good$\} \Rightarrow \{$sheet-percolation in $C_n\}$.

In order to prove Lemma 2, we shall need the following result.

Lemma 3. For $m<n$, let $F^m \subseteq J^{d,m}$ be a set of indices of level-m cubes such that $1_A[\mathbf{I}^{(m)}]=1$ and $1_Z[\mathbf{I}^{(m)}]=1$ for every $\mathbf{I}^{(m)} \in F^m$ and

$$S_m = \bigcup_{\mathbf{I}^{(m)} \in F^m} \{C[\mathbf{I}^{(m)}]\}$$

contains a full sheet separating $L=\{0\}\times[0,1]^{d-1}$ and $R=\{1\}\times[0,1]^{d-1}$. Then there exists $F^{m+1}\subseteq J^{d,m+1}$ such that $1_A[\mathbf{I}^{(m+1)}]=1$ and $1_Z[\mathbf{I}^{(m+1)}]=1$ for every $\mathbf{I}^{(m+1)}\in F^{m+1}$ and

$$S_{m+1} = \bigcup_{\mathbf{I}^{(m+1)} \in F^{m+1}} \{C[\mathbf{I}^{(m+1)}]\}$$

contains a full sheet separating L and R.

Proof of Lemma 3. We define the *core* $\tilde{S}_m$ of S_m by

$$\tilde{S}_m = \bigcup_{\mathbf{I}^{(m+1)} \in J^{d,m+1}} \{C[\mathbf{I}^{(m+1)}]: \operatorname{dist}(C[\mathbf{I}^{(m+1)}], [0,1]^d \backslash S_m) \geqslant \tfrac{2}{5} M^{-m}\}$$

so that $\tilde{S}_m$ is the union of those level-$(m+1)$ cubes which are distance at least $\frac{2}{5}M^{-m}$ from $[0,1]^d\backslash S_m$. We note that since S_m consists of cubes of side length M^{-m}, its core $\tilde{S}_m$ must also contain a full sheet separating L and R.

Pick an $\mathbf{I}^{(m+1)} \in J^{d,m+1}$ such that $C[\mathbf{I}^{(m+1)}] \subseteq \tilde{S}_m$; then we have

$$C[\mathbf{I}^{(m+1)}] \subseteq B[\mathbf{I}^{(m)}; \mathbf{k}] \subset \{A[\mathbf{I}^{(m)}; \mathbf{k}] \cup B[\mathbf{I}^{(m)}; \mathbf{k}]\} \subset C[\mathbf{I}^{(m)}] \subseteq S_m \tag{3.4}$$

for some $\mathbf{k}\in\{0,\dots,4\}^d$. Since $C[\mathbf{I}^{(m)}]$ consists of 5^d equal level-m blocks each of side length $\frac{1}{5}M^{-m}$, it is easy to see that we also have

$$\operatorname{dist}(B[\mathbf{I}^{(m)}; \mathbf{k}], [0,1]^d \backslash S_m) \geqslant \tfrac{2}{5} M^{-m} \tag{3.5}$$

All the level-m cubes contained in S_m are good, and hence we deduce from the definition of goodness that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ must be isolated.

We define

$$S_{m+1} = S_m \cap \bigcup_{\mathbf{I}^{(m+1)} \in J^{d,m+1}} \{C[\mathbf{I}^{(m+1)}] \colon 1_A[\mathbf{I}^{(m+1)}] = 1\}$$

and let F^{m+1} be the set of indices of the level-$(m+1)$ cubes contained in S_{m+1}. Pick $\mathbf{I}^{(m+1)} = (\mathbf{I}^{(m)}, \mathbf{i}_{m+1}) \in F^{m+1}$; we note that since $1_Z[\mathbf{I}^{(m)}] = 1$ and $Z[\mathbf{I}^{(m+1)}] = 1$ we have $1_Z[\mathbf{I}^{(m+1)}] = 1$ by (3.3).

Suppose that S_{m+1} does not contain a full sheet separating L and R, that is, there exists a chain $\Gamma = \{C(1),..., C(r)\}$ of level-$(m+1)$ cubes such that

$$\begin{aligned} C(j) &\nsubseteq S_{m+1} \qquad \text{for all} \quad 1 \leqslant j \leqslant r \\ C(1) \cap L &\neq \varnothing \\ C(r) \cap R &\neq \varnothing \\ C(j) \cap C(j+1) &\neq \varnothing \qquad \text{for all} \quad 1 \leqslant j < r \end{aligned} \tag{3.6}$$

Since $\tilde{S}_m$ does contain a full sheet separating L and R, we must have $C(i) \subseteq \tilde{S}_m$ for some $1 \leqslant i \leqslant r$; let $B[\mathbf{I}^{(m)}; \mathbf{k}]$ be the level-m block containing $C(i)$. By (3.4), we have $A[\mathbf{I}^{(m)}; \mathbf{k}] \subseteq S_m$, and hence we see that there is a chain $\Gamma' = \{C(s),..., C(t)\} \subseteq \Gamma$ of level-$(m+1)$ cubes such that

$$\begin{aligned} C(s) \cap \partial^{(i)} A[\mathbf{I}^{(m)}; \mathbf{k}] &\neq \varnothing \\ C(t) \cap \partial^{(o)} A[\mathbf{I}^{(m)}; \mathbf{k}] &\neq \varnothing \\ C(j) \cap C(j+1) &\neq \varnothing \qquad \text{for all} \quad s \leqslant j < t \end{aligned} \tag{3.7}$$

and $C(j)$ is not available for any $s \leqslant j \leqslant t$. But this means that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is not isolated, contradicting the above.

Hence we conclude that S_{m+1} does contain a full sheet separating L and R, as required. ∎

Proof of Lemma 2. Assume that $C[\varnothing]$ is good; then for every $\mathbf{k} \in \{0,..., 4\}^d$, the block $B[\varnothing; \mathbf{k}]$ is isolated. We let

$$F^1 = \{\mathbf{I}^{(1)} \in J^{d,1} \colon 1_A[\mathbf{I}^{(1)}] = 1\}$$

and so we see that the set

$$S_1 = \bigcup_{\mathbf{I}^{(1)} \in F^1} \{C[\mathbf{I}^{(1)}]\}$$

contains a full sheet separating L and R. We note that $1_Z[\mathbf{I}^{(1)}] = Z[\mathbf{I}^{(1)}] = 1$ for every $\mathbf{I}^{(1)} \in F^1$.

We now repeatedly apply Lemma 3 with $m = 1, 2,..., n-1$ to deduce that there exist sets $F^2, F^3,..., F^n$ such that $1_A[\mathbf{I}^{(m)}] = 1$ and $1_Z[\mathbf{I}^{(m)}] = 1$ for every $\mathbf{I}^{(m)} \in F^m$ and

$$S_m = \bigcup_{\mathbf{I}^{(m)} \in F^m} \{C[\mathbf{I}^{(m)}]\}$$

contains a full sheet separating L and R. In particular, when $m = n$, there exists

$$F^n \subseteq \{\mathbf{I}^{(n)} \in J^{d,n}: 1_Z[\mathbf{I}^{(n)}] = 1\}$$

such that

$$S_n = \bigcup_{\mathbf{I}^{(n)} \in F^n} \{C[\mathbf{I}^{(n)}]\}$$

contains a full sheet separating L and R. Since $S_n \subseteq C_n$, we see that sheet-percolation occurs in C_n, completing the proof of Lemma 2. ∎

We now consider site-percolation on the lattice $\mathbb{M}^d$, with sites being declared open independently with probability q. We let $\{0 \leftrightarrow \partial B(N)\}$ denote the event $\{$there exists an open path from the origin to a vertex of $\partial B(N)\}$, where

$$\partial B(N) = \{\mathbf{x} \in \mathbb{Z}^d: \max\{|x_i|: 1 \leqslant i \leqslant d\} = N\}$$

is the surface of the box of side length $2N$ centered at the origin.

Lemma 4. Suppose that $q < p_c(\mathbb{M}^d)$. Then there exists $\sigma(q) > 0$ such that for all N

$$P_q(0 \leftrightarrow \partial B(N)) \leqslant \exp[-N\sigma(q)]$$

This lemma, whose proof we omit, is a modified version of a result of Menshikov,[14] restated as Theorem 3.4 of Grimmett.[9] There it is given in the case of bond percolation on $\mathbb{Z}^d$, but the proof adapts readily to site percolation on $\mathbb{M}^d$.

We use Lemma 4 to estimate the probability that a level-m block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is isolated. Choose $\varepsilon > 0$ such that $(1-\varepsilon)p > 1 - p_c(\mathbb{M}^d)$, and define $\pi = (1-\varepsilon)p$. Let $m < n$ and suppose that each level-$(m+1)$ cube is available with probability π, independent of all other level-$(m+1)$ cubes. Let P_π denote the corresponding product probability measure on $\prod_{\mathbf{I} \in J^{d,m+1}} \{0, 1\}$. We compare C_{m+1} to a sublattice of $\mathbb{M}^d$ as follows: Open vertices of $B_{M^{m+1}}(\mathbb{M}^d)$ correspond to nonavailable cubes in C_{m+1}, with two vertices being considered adjacent if and only if the corresponding cubes have at least a point in common. Thus the existence of an open path from one of the vertices corresponding to a level-$(m+1)$ cube contained in $B[\mathbf{I}^{(m)}; \mathbf{k}]$ to the boundary of the box of side length $\frac{4}{5}M$ centered at this vertex will imply that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is not isolated.

Hence we deduce that

$$P_\pi(B[\mathbf{I}^{(m)}; \mathbf{k}] \text{ is not isolated}) \leqslant (\tfrac{1}{5}M)^d P_{1-\pi}(0 \leftrightarrow \partial B(\tfrac{2}{5}M))$$
$$\leqslant (\tfrac{1}{5}M)^d \exp[-\tfrac{2}{5}M\sigma(1-\pi)] \qquad (3.8)$$

by Lemma 4, since $1-\pi < p_c(\mathbb{M}^d)$. We shall take $M = M(\varepsilon)$ sufficiently large so that

$$M^d \exp[-\tfrac{2}{5}M\sigma(1-\pi)] < \varepsilon \qquad (3.9)$$

where $\varepsilon = 1 - \pi/p$.

For every $m \leqslant n$, we examine the probability that each level-m cube is good. Let $\mathscr{I}^{(n)} = J^{d,0} \cup J^{d,1} \cup \cdots \cup J^{d,n}$ be the set of all indices of cubes at level $\leqslant n$. For each $m \leqslant n$, let $\mathbf{I}^{(m)}(1), \mathbf{I}^{(m)}(2),..., \mathbf{I}^{(m)}(M^{dm})$ be some fixed ordering on $J^{d,m}$. We place an ordering on $\mathscr{I}^{(n)}$ as follows: For each $\mathbf{I}^{(m)} \in J^{d,m}$ and $\tilde{\mathbf{I}}^{(\tilde{m})} \in J^{d,\tilde{m}}$, where $0 \leqslant m, \tilde{m} \leqslant n$, we have $\mathbf{I}^{(m)} < \tilde{\mathbf{I}}^{(\tilde{m})}$ if and only if *either* $m > \tilde{m}$, *or* $m = \tilde{m}$ and $\mathbf{I}^{(m)}$ precedes $\tilde{\mathbf{I}}^{(\tilde{m})}$ in the ordering on $J^{d,m}$. The *initial segment* of $\mathscr{I}^{(n)}$ of length l is simply the set of the first l indices in the ordering on $\mathscr{I}^{(n)}$.

Suppose that we are given a family $\{X[\mathbf{I}]: \mathbf{I} \in \mathscr{I}^{(n)}\}$ of random variables, each $X[\mathbf{I}]$ taking values in $\{0, 1\}$. For each $\mathbf{I} \in \mathscr{I}^{(n)}$, let $\mathscr{F}(\mathbf{I}^-)$ be the σ-field in probability space generated by $\{X[\tilde{\mathbf{I}}]: \tilde{\mathbf{I}} < \mathbf{I}\}$. The appropriate sample space here is the product space $\Omega = \prod_{\mathbf{I} \in \mathscr{I}^{(n)}}\{0, 1\}$, points of which are represented as functions $\omega = (\omega(\mathbf{I}): \mathbf{I} \in \mathscr{I}^{(n)})$ on $\mathscr{I}^{(n)}$. The natural partial order on Ω is given by $\omega_1 \leqslant \omega_2$ if and only if $\omega_1(\mathbf{I}) \leqslant \omega_2(\mathbf{I})$ for all $\mathbf{I} \in \mathscr{I}^{(n)}$. We say that an event E on Ω is *increasing* if $\omega_1 \in E$ implies $\omega_2 \in E$ whenever $\omega_1 \leqslant \omega_2$.

Lemma 5. Suppose that we are given an initial segment $\mathscr{I}$ of $\mathscr{I}^{(n)}$ of length l and a family $\{X[\mathbf{I}]: \mathbf{I} \in \mathscr{I}\}$ of random variables each taking values in $\{0, 1\}$. Suppose that there exists $\rho \in [0, 1]$ such that

$$P(X[\mathbf{I}] = 1 \mid \mathscr{F}(\mathbf{I}^-)) \geqslant \rho$$

for all $\mathbf{I} \in \mathscr{I}$. Then for every increasing event E depending only on the outcomes $\{X[\mathbf{I}]: \mathbf{I} \in \mathscr{I}\}$, we have

$$P(E) \geqslant P_\rho(E)$$

where P_ρ is the product probability measure on $\prod_{\mathbf{I} \in \mathscr{I}} \{0, 1\}$ such that $X[\mathbf{I}] = 1$ with probability ρ for all $\mathbf{I} \in \mathscr{I}$.

This result appears as Lemma 3 of Falconer and Grimmett,[7] although no proof is given there. Also note that all of our conditional probability statements are in fact sure, rather than almost sure, since the sample space is finite.

Proof of Lemma 5. We proceed by induction on the length l of the initial segment of $\mathscr{I}^{(n)}$ under consideration. Let $\mathscr{I}^-$ denote the initial segment of $\mathscr{I}^{(n)}$ of length $(l-1)$, and assume that for every increasing event E^- on $\mathscr{I}^-$ we have

$$P(E^-) \geqslant P_\rho(E^-) \tag{3.10}$$

Observe that by the hypothesis of the lemma, we have

$$P(\{X[\mathbf{I}] = 1\} \cap B) \geqslant \rho P(B) \tag{3.11}$$

for every event B on $\mathscr{I}^-$.

Let E be an increasing event on $\mathscr{I}$; then we can write E as a disjoint union

$$E = \{E_1^- \cap \{X[\mathbf{I}] = 0\}\} \cup \{E_2^- \cap \{X[\mathbf{I}] = 1\}\} \tag{3.12}$$

where E_1^-, E_2^- are events on $\mathscr{I}^-$ and $\mathbf{I}$ is the lth cube in the ordering on $\mathscr{I}^{(n)}$. Since E is increasing, both E_1^- and E_2^- are also increasing; moreover, $E_1^- \subseteq E_2^-$. Then

$$E = E_1^- \cup \{(E_2^- \backslash E_1^-) \cap \{X[\mathbf{I}] = 1\}\} \tag{3.13}$$

and hence

$$\begin{aligned}
P(E) &= P(E_1^-) + P((E_2^- \backslash E_1^-) \cap \{X[\mathbf{I}] = 1\}) \\
&\geqslant P(E_1^-) + \rho P(E_2^- \backslash E_1^-) && \text{by (3.11)} \\
&= (1-\rho)\, P(E_1^-) + \rho P(E_2^-) \\
&\geqslant (1-\rho)\, P_\rho(E_1^-) + \rho P_\rho(E_2^-) && \text{by (3.10)} \\
&= P_\rho(E_1^-) + \rho P_\rho(E_2^- \backslash E_1^-) \\
&= P_\rho(E_1^-) + P_\rho((E_2^- \backslash E_1^-) \cap \{X[\mathbf{I}] = 1\}) && \text{since } P_\rho \text{ is Bernoulli} \\
&= P_\rho(E) && (3.14)
\end{aligned}$$

completing the proof. ∎

If A and B are both increasing events on $\mathscr{I}^{(n)}$ and P_ρ is the product probability measure as in Lemma 5, then we also have the FKG inequality:

$$P_\rho(A \cap B) \geqslant P_\rho(A)\, P_\rho(B) \tag{3.15}$$

This result is well known in percolation theory; it was first proved by Harris[10] and subsequently generalized by Fortuin *et al.*[8]

Lemma 6. If M is sufficiently large so that (3.9) holds, then for every $\mathbf{I} \in \mathscr{I}^{(n)}$

$$P(1_A[\mathbf{I}] = 1 \mid \mathscr{F}(\mathbf{I}^-)) \geqslant \pi$$

where $\mathscr{F}(\mathbf{I}^-)$ is the σ-field generated by $\{1_A[\tilde{\mathbf{I}}]: \tilde{\mathbf{I}} < \mathbf{I}\}$.

Proof of Lemma 6. We proceed by induction on the ordering on $\mathscr{I}^{(n)}$; for every $\mathbf{I} \in \mathscr{I}^{(n)}$, we suppose that

$$P(1_A[\tilde{\mathbf{I}}] = 1 \mid \mathscr{F}(\tilde{\mathbf{I}}^-)) \geqslant \pi \tag{3.16}$$

holds for all $\tilde{\mathbf{I}} < \mathbf{I}$, and show that we then also have

$$P(1_A[\mathbf{I}] = 1 \mid \mathscr{F}(\mathbf{I}^-)) \geqslant \pi \tag{3.17}$$

Certainly (3.17) is satisfied for the first M^{dn} terms in the ordering on $\mathscr{I}^{(n)}$, because all level-n cubes are good by definition and hence are available with probability $p > \pi$.

For $m < n$, we place an ordering $\mathbf{I}^{(m)}(1), \mathbf{I}^{(m)}(2),..., \mathbf{I}^{(m)}(M^{dm})$ on $J^{d,m}$ as before. Within each level-m cube $\mathbf{I}^{(m)}(j)$, $1 \leqslant j \leqslant M^{dm}$, we also order the indices $[\mathbf{I}^{(m)}(j), \mathbf{k}]$ of the 5^d blocks $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$, $\mathbf{k} \in \{0,..., 4\}^d$. Combining the two gives an ordering on the product space $J^{d,m} \times \{0,..., 4\}^d$ of the indices of all the level-m blocks: We say that $[\mathbf{I}^{(m)}, \mathbf{k}] < [\tilde{\mathbf{I}}^{(m)}, \tilde{\mathbf{k}}]$ if and only if *either* $\mathbf{I}^{(m)}$ precedes $\tilde{\mathbf{I}}^{(m)}$ in the ordering on $J^{d,m}$, *or* $\mathbf{I}^{(m)} = \tilde{\mathbf{I}}^{(m)}$ and $[\mathbf{I}^{(m)}, \mathbf{k}]$ precedes $[\tilde{\mathbf{I}}^{(m)}, \tilde{\mathbf{k}}]$. We label the level-$m$ blocks in order as $B(1), B(2),..., B(5^d M^{dm})$, and for $1 \leqslant l \leqslant 5^d M^{dm}$ let $\Delta(l)$ denote the event $\{B(l)$ is isolated$\}$. We observe that every $\Delta(l)$ is an increasing event on $J^{d,m+1}$.

We first examine the probability that the block $B(1) \subset C[\mathbf{I}^{(m)}(1)]$ is isolated. In this case, $\mathscr{F}(\mathbf{I}^{(m)}(1)^-)$ is the σ-field generated by $\{1_A[\tilde{\mathbf{I}}]: \tilde{\mathbf{I}} \in \mathscr{I}\}$, where $\mathscr{I} = J^{d,m+1} \cup \cdots \cup J^{d,n}$. Since $\Delta(1)$ is an increasing event on $J^{d,m+1} \subseteq \mathscr{I}$ and $\mathscr{I}$ is an initial segment of $\mathscr{I}^{(n)}$, we can apply Lemma 5 to the random variables $\{1_A[\mathbf{I}]: \mathbf{I} \in \mathscr{I}\}$ to give

$$\begin{aligned} P(\Delta(1)) &\geqslant P_\pi(\Delta(1)) \\ &\geqslant 1-(\tfrac{1}{5}M)^d \exp[-\tfrac{2}{5}M\sigma(1-\pi)] \\ &> 1-\varepsilon/5^d \end{aligned} \tag{3.18}$$

by (3.8), where P_π is the product probability measure as above.

Next we consider the lth block $B(l)$, $1 < l \leqslant 5^d M^{dm}$, in the ordering on the level-m blocks. In general, the event $\Delta(l)$ is not independent of $\{\Delta(k): k < l\}$. For each l, let

$$T(l) = \{k < l: d(B(l), B(k)) < \tfrac{2}{5}M^{-m}\}$$

and suppose that $\Delta(k)$ holds for every $k \in T(l)$. We note that $\bigcap_{k \in T(l)} \Delta(k)$ is also an increasing event on $J^{d,m+1}$ and hence we find that

$$\begin{aligned} P_\pi\left(\Delta(l) \,\middle|\, \bigcap_{k \in T(l)} \Delta(k)\right) &= \frac{P_\pi(\Delta(l) \cap \bigcap_{k \in T(l)} \Delta(k))}{P_\pi(\bigcap_{k \in T(l)} \Delta(k))} \\ &\geqslant P_\pi(\Delta(l)) \end{aligned} \tag{3.19}$$

by the FKG inequality (3.15).

For each $k < l$ such that $k \notin T(l)$, the annuli $A(l)$ and $A(k)$ around the blocks $B(l)$ and $B(k)$ have disjoint interiors, and hence have no level-$(m+1)$ cubes in common. Since we decide whether a block is isolated or not by examining solely the cubes contained in the annulus around that block, we see that under the product probability measure P_π the events $\Delta(l)$ and $\Delta(k)$ are independent. Similarly, $\Delta(l)$ is independent of any combination of events $\Delta(k)$ for such k, that is,

$$P_\pi(\Delta(l) \mid G) = P_\pi(\Delta(l)) \tag{3.20}$$

for every event $G \in \mathscr{G}(l)$, where $\mathscr{G}(l)$ is the σ-field generated by $\{\Delta(k): k < l, k \notin T(l)\}$. Combining (3.19) and (3.20), we find that

$$\begin{aligned} P_\pi\left(\Delta(l) \,\middle|\, \bigcap_{k \in T(l)} \Delta(k) \cap G\right) &\geqslant P_\pi(\Delta(l)) \\ &> 1-\varepsilon/5^d \end{aligned} \tag{3.21}$$

for every event by $G \in \mathscr{G}(l)$, by (3.8).

Next we wish to determine the goodness of each level-m cube $C[\mathbf{I}^{(m)}(j)]$, $1 \leqslant j \leqslant M^{dm}$, in order. To decide whether $C[\mathbf{I}^{(m)}(j)]$ is good or not, we need only examine the blocks $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$ such that

$$\operatorname{dist}(B[\mathbf{I}^{(m)}(j); \mathbf{k}], P(j)) \geqslant \tfrac{2}{5}M^{-m}$$

where

$$P(j) = \bigcup_{l<j} \{C[\mathbf{I}^{(m)}(l)]: C[\mathbf{I}^{(m)}(l)] \text{ is not good}\}$$

as before; let $N(j)$ denote the number of such blocks to be examined. We label these blocks as $B_1,..., B_{N(j)}$, where without loss of generality $B_1 < \cdots < B_{N(j)}$ are the first $N(j)$ blocks in the ordering on all the level-m blocks to be contained in $C[\mathbf{I}^{(m)}(j)]$. We let Δ_i denote the event $\{B_i$ is isolated$\}$. The cube $C[\mathbf{I}^{(m)}(j)]$ is good if and only if all the blocks $B_1,..., B_{N(j)}$ are isolated, and hence

$$\begin{aligned}
&P_\pi(C[\mathbf{I}^{(m)}(j)] \text{ is good} \mid \mathscr{F}(\mathbf{I}^{(m)}(j)^-)) \\
&= P_\pi\left(\bigcap_{i=1}^{N(j)} \Delta_i \,\middle|\, \mathscr{F}(\mathbf{I}^{(m)}(j)^-)\right) \\
&= P_\pi(\Delta_1 \mid \mathscr{F}(\mathbf{I}^{(m)}(j)^-)) \times P_\pi(\Delta_2 \mid \{\Delta_1, \mathscr{F}(\mathbf{I}^{(m)}(j)^-)\}) \\
&\quad \times \cdots \times P_\pi(\Delta_{N(j)} \mid \{\Delta_1,..., \Delta_{N(j)-1}, \mathscr{F}(\mathbf{I}^{(m)}(j)^-)\}) \\
&> (1-\varepsilon/5^d)^{N(j)} > 1-\varepsilon
\end{aligned} \tag{3.22}$$

since $N(j) \leqslant 5^d$, by applying (3.21) to each of the terms in the product. We note that the event $\{C[\mathbf{I}^{(m)}(j)]$ is good$\}$ is increasing on

$$\mathbf{I}^{(m)}(1) \cup \cdots \cup \mathbf{I}^{(m)}(j-1) \cup J^{d,m+1} \cup \cdots \cup J^{d,n}$$

and hence we can apply Lemma 5 to deduce that

$$\begin{aligned}
&P(C[\mathbf{I}^{(m)}(j)] \text{ is good} \mid \mathscr{F}(\mathbf{I}^{(m)}(j)^-)) \\
&\quad \geqslant P_\pi(C[\mathbf{I}^{(m)}(j)] \text{ is good} \mid \mathscr{F}(\mathbf{I}^{(m)}(j)^-)) \\
&\quad > 1-\varepsilon
\end{aligned} \tag{3.23}$$

by (3.22). Finally we note that

$$\{1_A[\mathbf{I}^{(m)}(j)] = 1\} = \{C[\mathbf{I}^{(m)}(j)] \text{ is good}\} \cap \{Z[\mathbf{I}^{(m)}(j)] = 1\} \tag{3.24}$$

and therefore

$$P(1_A[\mathbf{I}^{(m)}(j)] = 1 \mid \mathscr{F}(\mathbf{I}^{(m)}(j)^-)) > (1-\varepsilon)\, p = \pi \tag{3.25}$$

as required. ∎

To conclude the proof of Theorem 2, we take $M(\varepsilon)$ sufficiently large so that (3.9) holds for all $M \geqslant M(\varepsilon)$ and apply Lemmas 6 and 5 to deduce that

$$P(C[\varnothing] \text{ is good}) \geqslant P_\pi(C[\varnothing] \text{ is good}) \geqslant 1 - \varepsilon \tag{3.26}$$

The value of $M(\varepsilon)$ chosen works uniformly for all n, and hence by Lemma 2

$$P(\text{sheet-percolation in } C_n) \geqslant 1 - \varepsilon \tag{3.27}$$

for all $n \geqslant 1$, as required. ∎

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